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Simple Proofs of Two Results on Convolutions of Discrete Unimodal Distributions

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Abstract*

We present a much easier proofs of two results, given by Dharmadhikari and Joag-Dev about convolutions of discrete unimodal distributions. The first of these results states that the convolution of two symmetric unimodal distributions on is unimodal. The other result states that symmetrization of a unimodal random variable gives a symmetric unimodal random variable. These results are analogue of Wintner's theorem for symmetric continuous unimodal distributions. We use the method of representation of the expectation of a non-negative random variable with its tail probability, as employed in Purkayasha (1998), to prove these results.

Keywords and Phrases: Unimodality, Discrete Unimodality, Convolution, Symmetric, and Linear Expectation.

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* *We dedicate this work to our colleague Professor Mir Masoom Ali for his significant Contributions in the field of statistics.*

1 Introduction

We use the definition of discrete random variable X or its distribution function F being unimodal about a mode m, as it stated in Dharmadhikari and Joag-Dev (1988), Abouammoh and Mashhour (1994) and Ageel (1999, 2001, 2003). It is known that the property of unimodality is not preserved under convolution. Counterexamples demonstrating this can be found in Dharmadhikari and Joag-Dev (1988). However, a result on the convolutions of unimodal discrete distributions states that the convolution of two symmetric unimodal discrete distributions on l is unimodal. This result is due to Dharamadhikari and Joag-Dev (1988). Another result on convolutions states that symmetrizations of unimodal discrete distributions are unimodal. This result is also due to Dharmadhikari and Joag-Dev (1983). These two results showed that the symmetrization property could imply the unimodality of convolutions.

The original proofs of the above two results as it stated in Dharmadhikari and Joag-Dev (1983, 1988) are quite complicated. Our goal in this paper is to give a much easier proofs for both results using the method of representation of the expectation of non-negative random variable with its tail probability as given in Pukayasha (1998)

The organization of the paper is as follows. In section 2, we give some facts about unimodal distributions which are relevant in our context. Then the results are proved.

2 Proof of the results

We begin with a fact about unimdal distributions. Suppose F is the distribution function of a random variable X which is unimodal about some number m. Then the following is true (Dharmadhikari and Joag-Dev (1988); AlGhores, (2003)).

Proposition 2.1. Let F be a discrete distribution which can be written as

$$F(x) = p\delta_v(x) + (1-p)F_1(x), x \in I, \tag{1}$$

where $0 \le p \le 1$, δ_v is the distribution function of a random variable which is degenerate at v, and F_1 is distribution function which is unimodal about v.

Proposition 2.2. If X is a positive integer valued with a finite mean and with a decreasing (discrete) probability density function then X is skewed to the right.

Proposition 2.3. If f is symmetric about zero, then the unimodality index of f attains its minimum at the vertex zero.

Theorem 2.1. The convolution of two symmetric unimodal discrete distributions on I is unimodal.

Proof. Suppose F and G are two unimodal discrete distributions both symmetric around zero. Denote by f and g, the probability density functions corresponding to F and G respectively.

First note that the density of $F * G$ is given by

$$f * g(x) = \sum_{-\infty}^{\infty} f(y)g(x-y), x \in I$$

Notice that since both f and g are symmetric, so also is $f * g$ and hence we need only to show that $f * g$ is a decreasing function for $x \geq 0$.
Suppose that X is a random variable with density f, then

$$f * g(x) = E\left[g(x - X)\right] = E\left[g(X - x)\right]$$

the last equality being a consequence of the symmetry of g. Let us consider, for every $x \geq 0$, a random variable Y_x as

$$Y_x = g(X - x)$$

and then it follows

$$f * g(x) = \sum_{0}^{\infty} P(Y_x > u), u > 0$$

In view of our earlier proposition 2.2 it follows that $P(Y_x > u)$ is non-increasing function as a function of x and hence $f * g$ is non-increasing function for $x \geq 0$, i.e. attains its minimum at the vertex zero and this completes the proof of the theorem.
Theorem 2.2. Let X_1 and X_2 be two independent discrete random variables having the same unimodal distribution. Then $X_1 - X_2$ is unimodal.
Proof. Using assumptions in the proof of Theorem 2.1, clearly,

$$g(x) = f(-x), x \in I \tag{2}$$

Observe now that the distribution function of $X_1 - X_2$ is given by $F * G(x)$, with density given, in view of equation (2), by

$$f * g(x) = \sum_{-\infty}^{\infty} f(y)f(y-x)dy, x \in I$$

This is a symmetric density functin, and so it is enough to prove that $f * g$ is a decreasing function for $x \geq 0$. However, $f * g(x) = E\left[f(X - x)\right]$, where X is a random variable having density f. An argument similar to the one used in proving Theorem 2.1 can now be utilize to prove that $E\left[f(X - x)\right]$ is a decreasing function of x and this completes the proof of the theorem.

3 Conclusion

The discrete unimodality of a probability distributions is of considerable interest in a number of disciplines. Convolution is just one of the properties of α-unimodal probability distributions that one might study. This article has provided a very simple proofs for two existing α-unimodal convolution results. There are several unimodality results can be revisited and proved using different, new and much easier techniques and this article satisfied this aim and opened the door for further similar practice.

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References

1. Abouammoh, A. M. and Mashhour, A. F. (1994). Variance upper bounds and convolutions of α-unimodal distributions. *Statistics and Probability Letters*, 21, 281-289.

2. Ageel, M. I. A. (1999). On discrete α-unimodality. *Dirasat*, 26, 116-119.

3. Ageel, M. I. A. (2001). On characterizations of discrete α-unimodality: A survey. *International Journal of Applied Mathematics*, 5, 307-324.

4. Ageel, M. I. A. (2003). On convolutions of discrete α-unimodal distributions.*Journal of Applied Statistical Science*, 12(4), 297-304.

5. Ageel, M.I. A., Khurshid, A. and AlGhres, F.(2002). On convolutions of discrete unimodal distributions. *JSS, special volume, 275-282*

6. AlGhores, F. (2003). Some contributions to the discrete unimodality. *M. Sc Thesis, Girls College, Abha, Saudi Arabia.*

7. Dharmadhikari, S. W. and Joag-Dev, K. (1983). Unimodality of symmetrized unimodality laws and related results. In Bickel, P. J., Doksum, K. and Hodges, J. L. (Eds). *A Festschrift for Erich L. Lehmann.* Wadsworth, Belmont, California, pp. 131-139

8. Dharmadhikari, S. W. and Joag-Dev, K. (1988). *Unimodality, Convexity, and Applications.* Academic Press, New York.

9. Pukayasha, S. (1998). Simple proofs of two results on convolutions of unimodal distributions. *Statistics &Probability Letters, 39, 97-100.*

10. Wintner, A. (1938). Asymptotic distributions and infinite convolutions. *Edwards Brothers, Ann. Arbor, MI., USA*